Square Roots of Positive Operators

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Introduction

We provide a short, elementary proof of the existence and uniqueness of the square root in the context of bounded positive self-adjoint operators on real or complex Hilbert spaces.

Notations :

V	finite dimensional inner product space.
H	Hilbert space.
$\mathcal{L}(V)$	the set of all linear operators from a vector space V to itself.
$\mathcal{B}(H)$	the set of all bounded linear operators from H to itself.
ℓ_2	the space of all square-summable sequences.

Positive Operators

We know that if T is self-adjoint, $\langle Tx, x\rangle$ is real. Hence we may consider the set of all bounded self-adjoint linear operators on a complex Hilbert space H and introduce on this set a partial ordering $\leqq$ by defining

$$T_1 \leqq T_2 \qquad \text{if and only if} \qquad \langle T_1 x, x\rangle \leqq \langle T_2 x, x\rangle \tag{1}$$

for all $x \in H$. Instead of $T_1 \leqq T_2$ we also write $T_2 \geqq T_1$.

An important particular case is the following one. A bounded self-adjoint linear operator $T : H \longrightarrow H$ is said to be **positive**, written

$$T \geqq 0, \qquad \text{if and only if} \qquad \langle Tx, x\rangle \geqq 0 \;\text{ for all } x \in H. \tag{2}$$

Positive Operators

Instead of $T \geqq 0$ we also write $0 \leqq T$. Actually, such an operator should be called "nonnegative", but "positive" is the usual term.

Note the simple relation between (1) and (2), namely,

$$T_1 \leqq T_2 \qquad \Longleftrightarrow \qquad 0 \leqq T_2 - T_1,$$

that is, (1) holds if and only if $T_2 - T_1$ is positive.

We discuss positive operators and their square roots in the lecture, a topic which is interesting in itself and, moreover, will serve as a tool in the derivation of a spectral representation for bounded self-adjoint linear operators.

Positive Operators

The sum of positive operators is positive.

This is obvious from the definition. Let us turn to products. We know that a product (composite) of bounded self-adjoint linear operators is self-adjoint if and only if the operators commute, and we shall now see that in this case, positivity is preserved, too. This fact will be used quite often in our further work.

Theorem 1 (Product of positive operators).

If two bounded self-adjoint linear operators S and T on a Hilbert space H are positive and commute ($ST = TS$), then their product ST is positive.

Outline of the proof. We must show that $\langle STx, x\rangle \geqq 0$ for all $x \in H$. If $S = 0$, this holds. Let $S \neq 0$. We proceed in two steps (a) and (b):

(a) We consider

$$S_1 = \frac{1}{\|S\|} S, \qquad S_{n+1} = S_n - S_n^2 \qquad (n = 1, 2, \dots) \tag{3}$$

and prove by induction that

$$0 \leqq S_n \leqq I. \tag{4}$$

(b) We prove that $\langle STx, x\rangle \geqq 0$ for all $x \in H$.

The partial order relation defined by (2) also suggests the following concept.

Definition 2 (Monotone sequence).

A ***monotone sequence*** (T_n) *of self-adjoint linear operators* T_n *on a Hilbert space* H *is a sequence* (T_n) *which is either monotone increasing, that is,*

$$T_1 \leqq T_2 \leqq T_3 \leqq \dots$$

or monotone decreasing, that is,

$$T_1 \geqq T_2 \geqq T_3 \geqq \dots .$$

A monotone increasing sequence has the following remarkable property. (A similar theorem holds for a monotone decreasing sequence.)

Theorem 3 (Monotone Sequence Theorem).

Let (T_n) be a sequence of bounded self-adjoint linear operators on a complex Hilbert space H such that

$$T_1 \leqq T_2 \leqq \cdots \leqq T_n \leqq \cdots \leqq K \tag{5}$$

where K is a bounded self-adjoint linear operator on H. Suppose that any T_j commutes with K and with every T_m. Then (T_n) is strongly operator convergent ($T_n x \longrightarrow Tx$ for all $x \in H$) and the limit operator T is linear, bounded and self-adjoint and satisfies $T \leqq K$.

Outline of the proof. We consider $S_n = K - T_n$ and prove:

(a) The sequence $(\langle S_n^2 x, x \rangle)$ converges for every $x \in H$.

(b) $T_n x \longrightarrow Tx$, where T is linear and self-adjoint, and is bounded by the uniform boundedness theorem.

Problems

- Let S and T be bounded self-adjoint linear operators on a complex Hilbert space. If $S \leqq T$ and $S \geqq T$, show that $S = T$.
- Show that (1) defines a partial order relation on the set of all bounded self-adjoint linear operators on a complex Hilbert space H, and for any such operator T,

$$\begin{aligned} T_1 \leqq T_2 \quad &\Longrightarrow \quad T_1 + T \leqq T_2 + T \\ T_1 \leqq T_2 \quad &\Longrightarrow \quad \alpha T_1 \leqq \alpha T_2 \qquad (\alpha \geqq 0). \end{aligned}$$

Problems

- Let A, B, T be bounded self-adjoint linear operators on a complex Hilbert space H. If $T \geqq 0$ and commutes with A and B, show that

$$A \leqq B \qquad \text{implies} \qquad AT \leqq BT.$$

- If $T : H \longrightarrow H$ is a bounded linear operator on a complex Hilbert space H, show that TT^* and T^*T are self-adjoint and positive. Show that the spectra of TT^* and T^*T are real and cannot contain negative values. What are the consequences of the second statement for a square matrix A?
- Show that a bounded self-adjoint linear operator T on a complex Hilbert space H is positive if and only if its spectrum consists of nonnegative real values only. What does this imply for a matrix?

Problems

- Let $T : H \longrightarrow H$ and $W : H \longrightarrow H$ be bounded linear operators on a complex Hilbert space H and $S = W^* T W$. Show that if T is sel-adjoint and positive, so is S.
- Let T_1 and T_2 be bounded self-adjoint linear operators on a complex Hilbert space H and suppose that $T_1 T_2 = T_2 T_1$ and $T_2 \geqq 0$. Show that then $T_1^2 T_2$ is self-adjoint and positive.
- Let S and T be bounded self-adjoint linear operators on a Hilbert space H. If $S \geqq 0$, show that $TST \geqq 0$.
- Show that if $T \geqq 0$, then $(I + T)^{-1}$ exists.
- Let T be any bounded linear operator on a complex Hilbert space. Show that the inverse of $I + T^* T$ exists.

Problems

- Show that an illustrative example for Monotone Sequence Theorem is given by the sequence (P_n), where P_n is the projection of ℓ^2 onto the subspace consisting of all sequences $x = (\xi_j) \in \ell^2$ such that $\xi_j = 0$ for all $j > n$.
- If T is a bounded self-adjoint linear operator on a complex Hilbert space H, show that T^2 is positive. What does this imply for a matrix?
- If T is a bounded self-adjoint linear operator on a complex Hilbert space H, show that the spectrum of T^2 cannot contain a negative value. What theorem on matrices does this generalize?
- If $T : H \longrightarrow H$ and $S : H \longrightarrow H$ are bounded linear operators and T is compact and $S^*S \leqq T^*T$, show that S is compact.

Definition 4.

*Let V be a finite dimensional (real or complex) vector space. An operator $A \in \mathcal{L}(V)$ is a **square root** of an operator $T \in \mathcal{L}(V)$ if $A^2 = T$.*

In the following example and exercises, we consider real matrices.

Example 5.

If T is a rotation by the angle $\theta \in [0, 2\pi)$,

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

then A, a rotation by $\theta/2$, is a square root of T,

$$\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

Square Root

- Show that each of the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ does not have a square root matrices.
- Show that the matrix $A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ has only two square roots of A. Find them.
- Show that the identity matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has infinitely many square roots of A. Also, show that the identity operator on a finite dimensional vector space V has infinitely many square roots if $dim(V) > 1$.

Square Root

Theorem 6.

Let V be a complex finite dimensional vector space and $T \in \mathcal{L}(V)$ be invertible. Then T has a square root.

Note that in the above result the terms "complex" or "invertibility" cannot be dropped. Let us now recall the following result.

Theorem 7.

Let V be a complex finite dimensional inner prodct space and $T \in \mathcal{L}(V)$. Then T is self-adjoint if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in V$.

The above result is false for real inner product spaces as shown by considering any operator on a real inner product spaces that is not self-adjoint. **The result provides an example of how self-adjoint operators behave like real numbers.**

Positive Operator

Theorem 8.

*Let V be a finite dimensional inner prodct space. An operator $T \in \mathcal{L}(V)$ is called **positive** if T is self-adjoint and $\langle Tx, x \rangle \geq 0$, for all $x \in V$.*

Note that is V is a complex inner prodct space, then condition that T be self-adjoint can be dropped from this definition (by Theorem 7).

Many mathematicians also use the term positive semi-definite operator, which means the same as positive operator.

Positive Operator

We have seen that self-adjoint operators correspond, in some sense, to the real numbers. We shall now see that positive operators mimic the non-negative numbers.

Example 9.

Suppose $\alpha, \beta \in \mathbb{R}$ *and* $\alpha^2 < 4\beta$*. Let* x *be a real number. Then* $x^2 + \alpha x + \beta = (x + \frac{\alpha}{2})^2 + (\beta - \frac{\alpha^2}{4}) > 0.$

Proposition 10.

Let $T \in \mathcal{L}(V)$ *be self-adjoint and* $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha^2 < 4\beta$*, then* $T^2 + \alpha T + \beta I$ *is a positive operator.*

[Hint : $\langle (T^2 + \alpha T + \beta)x, x\rangle \geq \left(\|Tx\| - \frac{|\alpha|\|x\|}{2}\right)^2 + \left(\beta - \frac{\alpha^2}{4}\right)\|x\|^2 \geq 0$*, for all* $x \in V$*.]*

Theorem 11.

Let V be a finite dimensional inner prodct space and $T \in \mathcal{L}(V)$. Then the following are equivalent :

1. *T is positive;*
2. *T is self-adjoint and all the eigenvalues of T are non-negative;*
3. *T has a positive square root;*
4. *T has a self-adjoint square root;*
5. *there exists an operator $S \in \mathcal{L}(V)$ such that $T = S^*S$.*

We shall now discuss square roots for bounded operators on Hilbert spaces.

If T is self-adjoint, then T^2 is positive since $\langle T^2x, x\rangle = \langle Tx, Tx\rangle \geqq 0$, for all $x \in H$. We consider the converse problem: given a positive operator T, find a self-adjoint A such that $A^2 = T$. This suggests the following concept, which will be basic in connection with spectral representations.

Definition 12 (Positive square root).

*Let $T : H \longrightarrow H$ be a positive bounded self-adjoint linear operator on a complex Hilbert space H. Then a bounded self-adjoint linear operator A is called a **square root** of T if*

$$A^2 = T. \tag{6}$$

*If, in addition, $A \geqq 0$, then A is called a **positive square root** of T.*

Square Roots of a Positive Operator

Each non-negative number has a unique non-negative square root. The next theorem shows that positive operators enjoy a similar property. Because of this result, we can use the notation $\sqrt{T}$ to denote the unique positive square root of a positive operator T, just as $\sqrt{\lambda}$ denotes the unique non-negative square root of a non-negative number λ.

A positive operator can have ininitely many square roots (though only one of them can be positive).

For a positive $T \in \mathcal{B}(H)$, a positive square of T exists and is unique:

Theorem 13 (Positive square root).

Every positive bounded self-adjoint linear operator $T : H \longrightarrow H$ on a complex Hilbert space H has a positive square root A, which is unique. This operator A commutes with every bounded linear operator on H which commutes with T.

Outline of the proof. We proceed in three steps:

(a) We show that if the theorem holds under the additional assumption $T \leqq I$, it also holds without that assumption.

(b) We obtain the existence of the operator $A = T^{1/2}$ from $A_n x \longrightarrow Ax$, where $A_0 = 0$ and

$$A_{n+1} = A_n + \frac{1}{2}(T - A_n^2), \qquad n = 0, 1, \dots, \tag{7}$$

and we also prove the commutativity stated in the theorem.

(c) We prove uniqueness of the positive square root.

Square Roots of a Positive Operator

Square roots will play a basic role in connection with the spectral representation of bounded self-adjoint linear operators which will be discussed later.

Problems

- Find operators $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $T^2 = I$, the identity operator. Indicate which of the square roots is the positive square root of I.
- Let $T : L^2[0,1] \longrightarrow L^2[0,1]$ be defined by $(Tx)(t) = tx(t)$. Show that T is self-adjoint and positive and find its positive square root.
- Let $T : \ell^2 \longrightarrow \ell^2$ be defined by $(\xi_1, \xi_2, \xi_3, \dots) \longmapsto (0, 0, \xi_3, \xi_4, \dots)$. Is T bounded? Self-adjoint? Positive? Find a square root of T.
- Show that for the square root in (Positive Square Root) Theorem we have
$$\|T^{1/2}\| = \|T\|^{1/2}.$$

Problems

- Let $T : H \longrightarrow H$ be a bounded positive self-adjoint linear operator on a complex Hilbert space. Using the positive square root of T, show that for all $x, y \in H$,

$$|\langle Tx, y\rangle| \leqq \langle Tx, x\rangle^{1/2} \langle Ty, y\rangle^{1/2}.$$

  (a) It is interesting to note that the above statement can also be proved without the use of $T^{1/2}$. Give such a proof (which is similar to that of the Schwarz inequality).

  (b) Show that for all $x \in H$,

$$\|Tx\| \leqq \|T\|^{1/2} \langle Tx, x\rangle^{1/2},$$

  so that $\langle Tx, x\rangle = 0$ if and only if $Tx = 0$.

Problems

- Let B be a nonsingular n-rowed real square matrix and $C = BB^T$.
  (a) Show that C has a nonsingular positive square root A.
  (b) Show that $D = A^{-1}B$ with A and B given above is an orthogonal matrix.
- If S and T are positive bounded self-adjoint linear operators on a complex Hilbert space H and $S^2 = T^2$, show that $S = T$.

References

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- Kreyszig, Erwin, *Introductory Functional Analysis with Applications*, John Wiley & Sons, 1978.